

A BIJECTIVE PROOF OF VERSHIK'S RELATIONS FOR THE KOSTKA NUMBERS

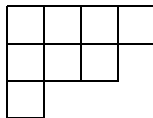
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ABSTRACT. We give a bijective proof of Vershik's relations for the Kostka numbers using insertion and reverse insertion algorithms.

1. INTRODUCTION

Throughout this paper, n will denote a positive integer. We write $\lambda \models n$ if λ is a composition of n , that is, a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ of nonnegative integers such that $|\lambda| = \sum_{i=1}^h \lambda_i = n$. In particular, if a sequence λ is non-increasing and $\lambda_i > 0$ for all $1 \leq i \leq h$, then we write $\lambda \vdash n$ and say that λ is a partition of n .

Let $\mu \vdash n$ and $\lambda \models n$. We denote by $h(\lambda)$ the height of λ , and by D_μ the Young diagram of μ . For example, let $\mu = (4, 3, 1) \vdash 8$. Then the Young diagram D_μ is



The rows and the columns are numbered from top to bottom and from left to right, like the rows and the columns of a matrix, respectively. A semistandard Young tableau (SSYT) of shape μ and weight, or content, λ is a filling of the Young diagram D_μ with the numbers $1, 2, \dots, h(\lambda)$ in such a way that

- (i) i occupies λ_i boxes, for $i = 1, 2, \dots, h(\lambda)$,
- (ii) the numbers are strictly increasing down the columns and weakly increasing along the rows.

We denote by $\text{SSYT}(\mu, \lambda)$ the set of all semistandard tableaux of shape μ and weight λ . In particular, if weight $\lambda = (1^n)$, then such a tableau is called a standard Young tableau (SYT) of shape μ . The Kostka number $K(\mu, \lambda)$ is defined to be the cardinality of $\text{SSYT}(\mu, \lambda)$.

We denote by $\tilde{\lambda}$ the partition obtained by rearranging components of a composition λ , and by $\lambda^{(i)}$ the composition of $n-1$ defined by $\lambda_i^{(i)} = \lambda_i - 1$, and $\lambda_j^{(i)} = \lambda_j$ otherwise. For $\lambda = (\lambda_1, \dots, \lambda_h) \vdash n$ and $\gamma \vdash n-1$, we write $\gamma \preceq \lambda$ if $\gamma_i \leq \lambda_i$ for all i with $1 \leq i \leq h$, and define

$$C(\lambda, \gamma) = |\{i \mid 1 \leq i \leq h, \tilde{\lambda}^{(i)} = \gamma\}|.$$

Vershik's relations for the Kostka numbers is as follows:

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Theorem 1 ([1, p.143, Theorem 3.6.13] and [5, Theorem 4]). *For any $\lambda \vdash n$ and $\rho \vdash n-1$, we have*

$$\sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) = \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).$$

Theorem 1 arises from restricting a permutation representation of the symmetric group \mathfrak{S}_n to \mathfrak{S}_{n-1} and then applying Young's rule to both sides. As previously stated, since $K(\mu, \lambda) = |\text{SSYT}(\mu, \lambda)|$, it is natural to expect a bijective proof of Theorem 1. In fact, Vershik [5, Theorem 4] claims to give a bijection from

$$\mathcal{L} = \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \text{SSYT}(\mu, \lambda)$$

to

$$\mathcal{R} = \bigcup_{1 \leq x \leq h} \text{SSYT}(\rho, \lambda^{(x)}),$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \vdash n$ and $\rho \vdash n-1$. In order to explain his proof, we call a tableau in $\text{SSYT}(\mu, \lambda)$ a μ -tableau, and a tableau in $\text{SSYT}(\rho, \lambda^{(x)})$ a ρ -tableau. Since μ -tableaux have one more box than ρ -tableaux, Vershik [5, Theorem 4] claims that removable of one box from μ -tableaux gives a bijection from \mathcal{L} to \mathcal{R} . Vershik [5, Section 4] gives examples, each of which comes with a bijection. However, if $\lambda = (3, 3, 2) \vdash 8$ and $\rho = (4, 3) \vdash 7$ then there is no bijection from \mathcal{L} to \mathcal{R} arising from removable of one box. More precisely, we consider two tableaux in \mathcal{L} as follows:

$$A = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 2 & & \\ \hline \end{array}, \quad E = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \\ \hline 3 & & & \\ \hline \end{array}.$$

The only ρ -tableau obtainable from A by removing one box is

$$Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \\ \hline \end{array}.$$

Similarly, the only ρ -tableau obtainable from E by removing one box is Q .

Let $[h] = \{1, 2, \dots, h\}$, and let $\text{SSYT}_{[h]}(\mu)$ be the set of all SSYT's of shape μ and taking values in $[h]$. For a partition $\rho \vdash n-1$, Loehr [3, p.399, 10.60] shows that insertion I and reverse insertion R give mutually inverse bijections

$$I : \text{SSYT}_{[h]}(\rho) \times [h] \rightarrow \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \text{SSYT}_{[h]}(\mu),$$

$$R : \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \text{SSYT}_{[h]}(\mu) \rightarrow \text{SSYT}_{[h]}(\rho) \times [h].$$

given by $I(T, x) = T \leftarrow x$ and $R(S)$ is the result of applying reverse insertion to S starting at the unique box of S not in ρ . In this paper, we describe a bijection between \mathcal{R} and \mathcal{L} using the restriction $I|_{\mathcal{R}'}$ and $R|_{\mathcal{L}}$, where $\mathcal{R}' = \bigcup_{1 \leq x \leq h} (\text{SSYT}(\rho, \lambda^{(x)}) \times \{x\})$.

This paper is organized as follows. After giving preliminaries in Section 2, we prove Theorem 1 in Section 3. In Section 4, another bijective proof of Vershik's relations can

be given using the Robinson-Schensted-Knuth correspondence. Finally, in section 5, we give two examples and compare Vershik's bijection with ours.

2. PRELIMILARIES

Throughout this paper, let $h \geq 1$, $x \geq 1$ and $n \geq 1$ be integers. Let $\mu \vdash n$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \models n$, and $T \in \text{SSYT}(\mu, \lambda)$. First of all, we need a fundamental combinatorial algorithm on tableaux called *row-insertion*, or *bumping* (see [2, Chapter 1]). We define a insertion tableau, denoted $T \leftarrow x$, by the following procedure.

Algorithm 1. Input: Let $T \in \text{SSYT}(\mu, \lambda)$ and x be a positive integer.

Output: $T \leftarrow x$

Initialization: $S := T$, $y := x$ and $i := 1$

while $|\{j \mid y < S(i, j)\}| > 0$ **do**

$z := \min\{j \mid y < S(i, j)\}$.

$x' := S(i, z)$.

if $(p, q) = (i, z)$ **then** $U(p, q) := y$

else

$U(p, q) := S(p, q)$

end if

$S \leftarrow U$, $y \leftarrow x'$ and $i \leftarrow i + 1$.

end

$T \leftarrow x(i, \mu_i + 1) := y$

Otherwise, $T \leftarrow x(p, q) := S(p, q)$

Output $T \leftarrow x$.

For example, let $\mu = (4, 3, 2, 1, 1) \vdash 11$ and $\lambda = (1, 3, 2, 2, 1, 2) \models 11$. Suppose $x = 2$ and

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 4 & 4 & \\ \hline 3 & 6 & & \\ \hline 5 & & & \\ \hline 6 & & & \\ \hline \end{array}.$$

The $x = 2$ bumps the 3 from the first row, which then bumps the first 4 from the second row, which bumps the 6 from the third row, which can be put at the end of the fourth row. Then

$$T \leftarrow x = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 4 & \\ \hline 3 & 4 & & \\ \hline 5 & 6 & & \\ \hline 6 & & & \\ \hline \end{array}.$$

It is clear that $T \leftarrow x$ is a SSYT of shape $\nu = (4, 3, 2, 2, 1) \vdash 12$ and weight $(1, 4, 2, 2, 1, 2) \models 12$, where $\mu \preceq \nu$.

This algorithm is invertible. Given a partition $\mu \vdash n$ and insertion tableau $T \leftarrow x$, there is the unique box of $T \leftarrow x$ not in D_μ (In above example, it is $\{(4, 2)\}$). From this box, we can construct the reverse insertion algorithm, so we can recover the original tableau T .

3. VERSHIK'S RELATIONS FOR THE KOSTKA NUMBERS

Theorem 2. *Let $\rho \vdash n-1$ and $\lambda = (\lambda_1, \dots, \lambda_h) \models n$ and, set*

$$\begin{aligned}\mathcal{R}' &= \bigcup_{1 \leq x \leq h} (\text{SSYT}(\rho, \lambda^{(x)}) \times \{x\}), \\ \mathcal{L} &= \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \text{SSYT}(\mu, \lambda).\end{aligned}$$

Then $I|_{\mathcal{R}'}$ and $R|_{\mathcal{L}}$ give mutually inverse bijections between \mathcal{R}' and \mathcal{L} .

Proof. It is obvious that

$$\begin{aligned}\mathcal{R}' &\subset \text{SSYT}_{[h]}(\rho) \times [h], \\ \mathcal{L} &\subset \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \text{SSYT}_{[h]}(\mu).\end{aligned}$$

For each $x \in [h]$, we have

$$I(\text{SSYT}(\rho, \lambda^{(x)}) \times \{x\}) \subset \mathcal{L}$$

by the definition of insertion. This implies $I(\mathcal{R}') \subset \mathcal{L}$. Conversely, for each $\mu \vdash n$ with $\mu \succeq \rho$, there exists ℓ such that $D_\mu = D_\rho \cup \{(\ell, \mu_\ell)\}$. Applying reverse insertion at (ℓ, μ_ℓ) for each tableau in $\text{SSYT}(\mu, \lambda)$, we find $R(\text{SSYT}(\mu, \lambda)) \subset \mathcal{R}'$. This implies $R(\mathcal{L}) \subset \mathcal{R}'$. Since I and R are mutually inverse bijections, we have

$$\begin{aligned}\mathcal{R}' &= RI(\mathcal{R}') \subset R(\mathcal{L}), \\ \mathcal{L} &= IR(\mathcal{L}) \subset I(\mathcal{R}').\end{aligned}$$

Therefore, $I(\mathcal{R}') = \mathcal{L}$ and $R(\mathcal{L}) = \mathcal{R}'$. □

From Theorem 2, we also construct a bijection from \mathcal{R} to \mathcal{L} given by $T \mapsto T \leftarrow x$.

Corollary 3 (Vershik's relations for the Kostka numbers). *For $\rho \vdash n-1$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \models n$, we have*

$$\sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) = \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).$$

Proof.

$$\begin{aligned}\sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) &= \sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} |\text{SSYT}(\mu, \lambda)| \\ &= \sum_{1 \leq x \leq h} |\text{SSYT}(\rho, \lambda^{(x)})| && \text{(by Theorem 2)} \\ &= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} \sum_{\substack{1 \leq x \leq h \\ \lambda^{(x)} = \gamma}} |\text{SSYT}(\rho, \lambda^{(x)})| \\ &= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} \sum_{\substack{1 \leq x \leq h \\ \lambda^{(x)} = \gamma}} |\text{SSYT}(\rho, \gamma)| && \text{(by [1, Lemma 3.7.1])}\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} |\{x \mid 1 \leq x \leq h, \widetilde{\lambda^{(x)}} = \gamma\}| |\text{SSYT}(\rho, \gamma)| \\
&= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).
\end{aligned}$$

□

4. THE ROBINSON-SCHENSTED-KNUTH CORRESPONDENCE

Another bijective proof of Vershik's relations can be given using the Robinson-Schensted-Knuth (RSK) correspondence (see [2, Chapter 4] and [4, Section 4.8]).

Definition 4 ([4, Definition 4.8.1]). A *generalized permutation* is a two-line array of positive integers

$$\pi = \begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & j_m \end{pmatrix}$$

whose columns are in lexicographic order, with the top entry taking precedence. Let $\hat{\pi}$ and $\tilde{\pi}$ stand for the top and bottom rows of π , respectively. We denote by $\text{cont } \hat{\pi}$ and $\text{cont } \tilde{\pi}$ the composition, where the i -th component equals the number of i 's in $\hat{\pi}$ and $\tilde{\pi}$, respectively.

For example, let

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 1 & 2 & 1 \end{pmatrix}$$

be a generalized permutation, $\text{cont } \hat{\pi} = (3, 2, 1)$ and $\text{cont } \tilde{\pi} = (2, 2, 2)$.

For $\nu \models n$ and $\lambda \models n$, let $\text{GP}(\nu, \lambda)$ be the set of all generalized permutations π such that $\text{cont } \hat{\pi} = \nu$ and $\text{cont } \tilde{\pi} = \lambda$. The RSK correspondence is a bijection

$$\text{RSK} : \text{GP}(\nu, \lambda) \rightarrow \bigcup_{\mu \vdash n} \text{SSYT}(\mu, \lambda) \times \text{SSYT}(\mu, \nu)$$

given by $\text{RSK}(\pi) = (P(\pi), Q(\pi))$, where $P(\pi)$ and $Q(\pi)$ is the insertion tableau and the recoding tableau for π , respectively. We consider a bijection

$$\Phi : \text{GP}((1^n), \lambda) \rightarrow \bigcup_{1 \leq x \leq h(\lambda)} \text{GP}((1^{n-1}), \lambda^{(x)})$$

given by

$$\Phi\left(\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ j_1 & j_2 & \cdots & j_{n-1} & x \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ j_1 & j_2 & \cdots & j_{n-1} \end{pmatrix}.$$

We define Ψ by the following diagram:

$$\begin{array}{ccc}
\text{GP}((1^n), \lambda) & \xrightarrow{\text{RSK}} & \bigcup_{\mu \vdash n} \text{SSYT}(\mu, \lambda) \times \text{SYT}(\mu) \\
\downarrow \Phi & & \uparrow \Psi \\
\bigcup_{1 \leq x \leq h(\lambda)} \text{GP}((1^{n-1}), \lambda^{(x)}) & \xrightarrow{\text{RSK}} & \bigcup_{1 \leq x \leq h(\lambda)} \bigcup_{\rho \vdash n-1} \text{SSYT}(\rho, \lambda^{(x)}) \times \text{SYT}(\rho)
\end{array}$$

For $\mu \vdash n$ and $\rho \vdash n-1$ with $\rho \preceq \mu$, we let $\mathcal{T}(\mu, \rho)$ be the set of all $T \in \text{SYT}(\mu)$ such that

$$T(i, j) = \begin{cases} n & \text{if } D_\mu \setminus D_\rho = \{(i, j)\}, \\ T'(i, j) & \text{otherwise} \end{cases}$$

for some $T' \in \text{SYT}(\rho)$.

Let $\rho \vdash n-1$ and $\lambda \models n$ and, set

$$\begin{aligned} \mathcal{R}'' &= \bigcup_{1 \leq x \leq h(\lambda)} \text{SSYT}(\rho, \lambda^{(x)}) \times \text{SYT}(\rho), \\ \mathcal{L}' &= \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \text{SSYT}(\mu, \lambda) \times \mathcal{T}(\mu, \rho). \end{aligned}$$

Now, we claim that Ψ maps \mathcal{R}'' to \mathcal{L}' . Indeed, if $(P', Q') \in \mathcal{R}''$, then

$$(P', Q') \mapsto \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ j_1 & j_2 & \cdots & j_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ j_1 & j_2 & \cdots & j_{n-1} & x \end{pmatrix} \mapsto (P, Q),$$

where $P = P' \leftarrow x$ and $Q(i, j) = n$ if $D_\mu \setminus D_\rho = \{(i, j)\}$, and $Q(i, j) = Q'(i, j)$ otherwise. Thus $\Psi(\mathcal{R}'') \subset \mathcal{L}'$. Conversely, if $(S, T) \in \mathcal{L}'$, from the definition of $\mathcal{T}(\mu, \rho)$, there is a box $\{(i, j)\} = D_\mu \setminus D_\rho$, so there exists $(P', Q') \in \mathcal{R}''$ such that $\Psi((P', Q')) = (S, T)$. This proves $\Psi(\mathcal{R}'') = \mathcal{L}'$. Since

$$\begin{aligned} |\mathcal{R}''| &= \sum_{1 \leq x \leq h(\lambda)} K(\rho, \lambda^{(x)}) \cdot |\text{SYT}(\rho)|, \\ |\mathcal{L}'| &= \sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) \cdot |\mathcal{T}(\mu, \rho)|, \end{aligned}$$

and $|\mathcal{T}(\mu, \rho)| = |\text{SYT}(\rho)|$, we obtain another bijective proof of Corollary 3.

5. EXAMPLES

In this section, we compare Vershik's bijection with ours.

Example 5 ([5, Example 1]). Let $\lambda = (3, 2, 1) \vdash 6$ and $\rho = (4, 1) \vdash 5$. Then

$$\begin{aligned} \mu\text{-tableaux : } A &= \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & & & & \\ \hline \end{array}, \quad B = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & & & & \\ \hline \end{array}, \quad C = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & & \\ \hline \end{array}, \\ D &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & & \\ \hline \end{array}, \quad E = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & & & \\ \hline 3 & & & \\ \hline \end{array}; \\ \rho\text{-tableaux : } L &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 3 & & & \\ \hline \end{array}, \quad M = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & & & \\ \hline \end{array}, \quad N = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 3 & & & \\ \hline \end{array}, \\ P &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & & & \\ \hline \end{array}, \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & & & \\ \hline \end{array}. \end{aligned}$$

We remove one box from the first row in A and B , one box from the second row in C and D , and one box $(3, 1)$ in E in order to obtain ρ -tableaux. Then we have a bijection as follows:

$$A \leftrightarrow L; \quad B \leftrightarrow M; \quad C \leftrightarrow N; \quad D \leftrightarrow P; \quad E \leftrightarrow Q.$$

The bijection given by Theorem 2 is:

$$\begin{aligned} L &\leftrightarrow L \leftarrow 1 = E; & M &\leftrightarrow M \leftarrow 1 = D; \\ N &\leftrightarrow N \leftarrow 2 = A; & P &\leftrightarrow P \leftarrow 2 = C; \\ Q &\leftrightarrow Q \leftarrow 3 = B. \end{aligned}$$

We give an example, for which there is no bijection arising from removable of one box.

Example 6. Let $\lambda = (3, 3, 2) \vdash 8$ and $\rho = (4, 3) \vdash 7$. Then

$$\begin{aligned} \mu\text{-tableaux : } A &= \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 2 & & \\ \hline \end{array}, B = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & & \\ \hline \end{array}, C = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & & \\ \hline \end{array}, \\ D &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline \end{array}, E = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \\ \hline 3 & & & \\ \hline \end{array}, F = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \hline 3 & & & \\ \hline \end{array}; \\ \rho\text{-tableaux : } L &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & \\ \hline \end{array}, M = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & \\ \hline \end{array}, N = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 3 & \\ \hline \end{array}, \\ P &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & \\ \hline \end{array}, Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \\ \hline \end{array}, R = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \hline \end{array}. \end{aligned}$$

As mentioned in Section 1, μ -tableaux A and E result in ρ -tableau Q , so there is no bijection between μ -tableaux and ρ -tableaux arising from removable of one box. The bijection given by Theorem 2 is:

$$\begin{aligned} L &\leftrightarrow L \leftarrow 1 = E; & M &\leftrightarrow M \leftarrow 1 = F; \\ N &\leftrightarrow N \leftarrow 2 = D; & P &\leftrightarrow P \leftarrow 2 = C; \\ Q &\leftrightarrow Q \leftarrow 3 = A; & R &\leftrightarrow R \leftarrow 3 = B. \end{aligned}$$

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